Probability Review

Why Probabilistic Robotics?

- autonomous mobile robots need to accommodate the uncertainty that exists in the physical world
- sources of uncertainty
  - environment
  - sensors
  - actuation
  - software
  - algorithmic
- probabilistic robotics attempts to represent uncertainty using the calculus of probability theory

Axioms of Probability Theory

Pr(*A*) denotes probability that proposition *A* is true.

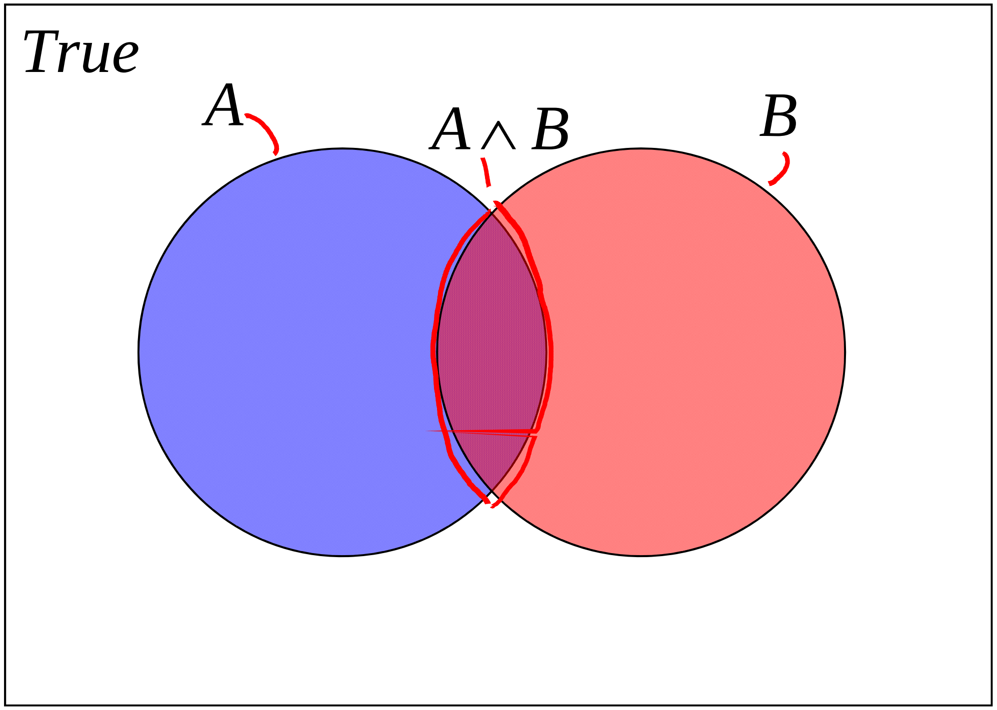
- $0 \leq \Pr(A) \leq 1$
- $\Pr(True) = 1 \qquad \Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

OR AND

A Closer Look at Axiom 3

OR   AND

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Using the Axioms

NOT

Axiom 3

$$
\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}
$$

Discrete Random Variables

- X denotes a random variable.
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable X takes on value x_i.
- $P(\cdot)$ is called probability mass function.

Discrete Random Variables

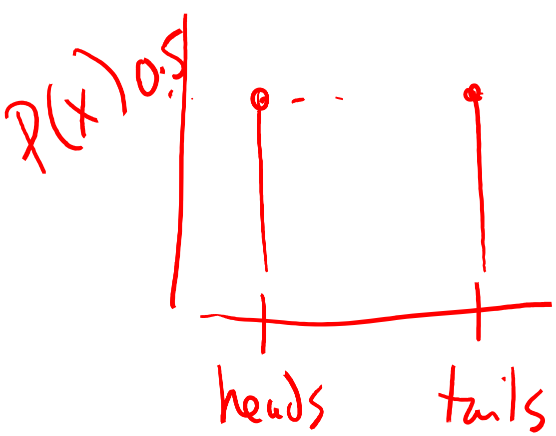
- fair coin

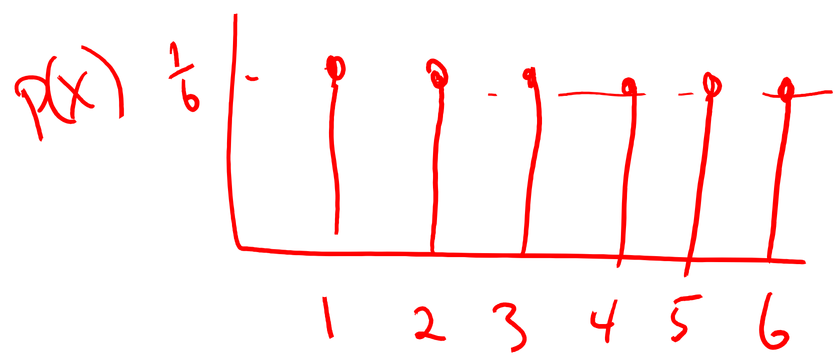
$$P(X=heads) = P(X=tails) = 1/2$$

sum of all probabilities = 1

- fair dice

$$P(X=1) = P(X=2) = P(X=3) = P(X=4) = P(X=5) = P(X=6) = 1/6$$





Discrete Random Variables

- sum of two fair dice } – a random variable

P(X=2)	(1,1)	1/36
P(X=3)	(1,2), ~~(2,3)~~ (2,1)	2/36
P(X=4)	(1,3), (2,2), (3,1)	3/36
P(X=5)	(1,4), (2,3), (3,2), (4,1)	4/36
P(X=6)	(1,5), (2,4), (3,3), (4,2), (5,1)	5/36
P(X=7)	(1,6), (2,5), (3,4), (4,3), (5,2), (6, 1)	6/36
P(X=8)	(2, 6), (3, 5), (4,4), (5,3), (6, 2)	5/36
P(X=9)	(3, 6), (4, 5), (5, 4), (6, 3)	4/36
P(X=10)	(4, 6), (5, 5), (6, 4)	3/36
P(X=11)	(5, 6), (6, 5)	2/36
P(X=12)	(6, 6)	1/36

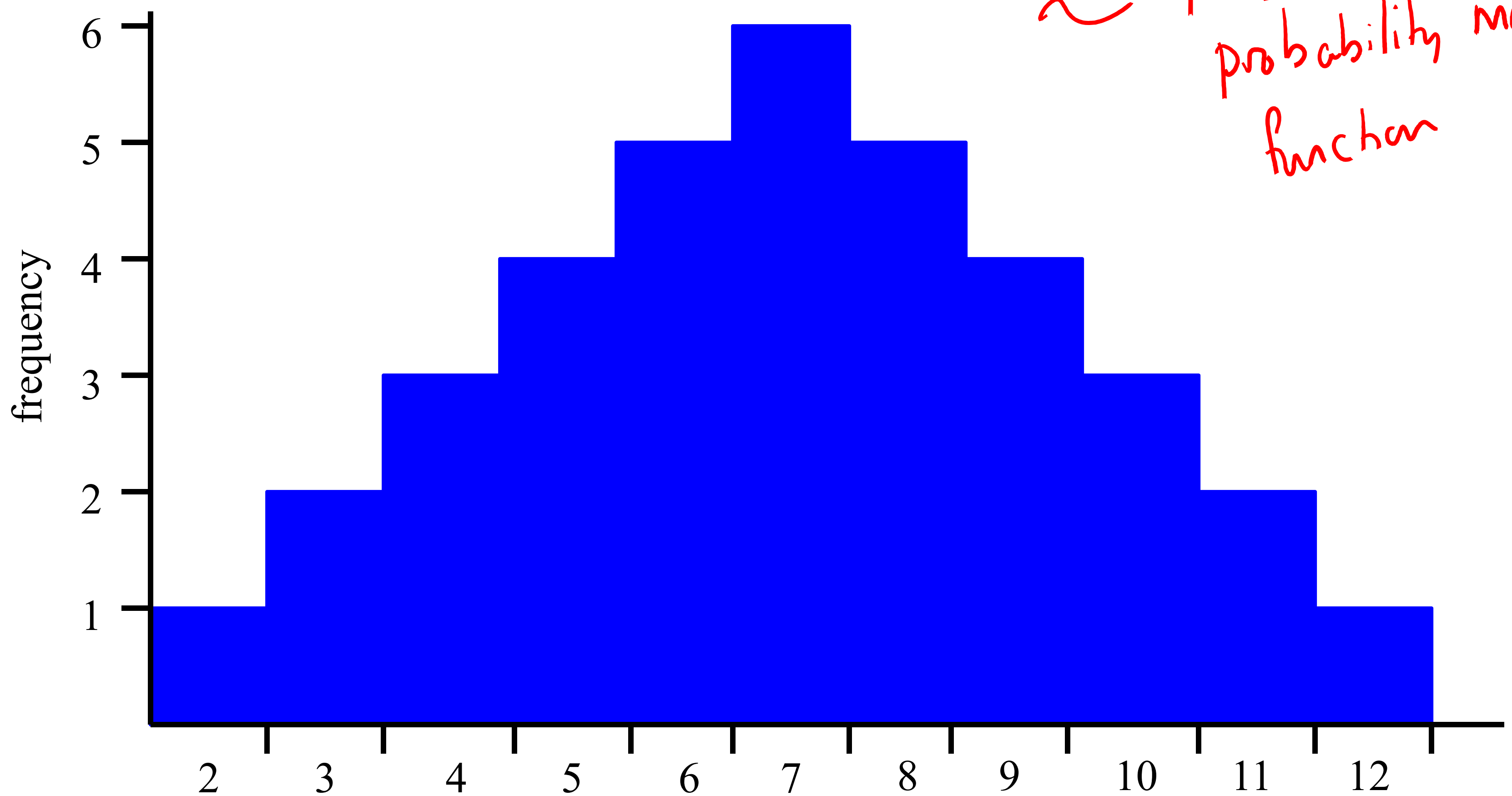
} probability mass function

sum = 1

Discrete Random Variables

- plotting the frequency of each possible value yields the histogram

~ proportional to probability mass function



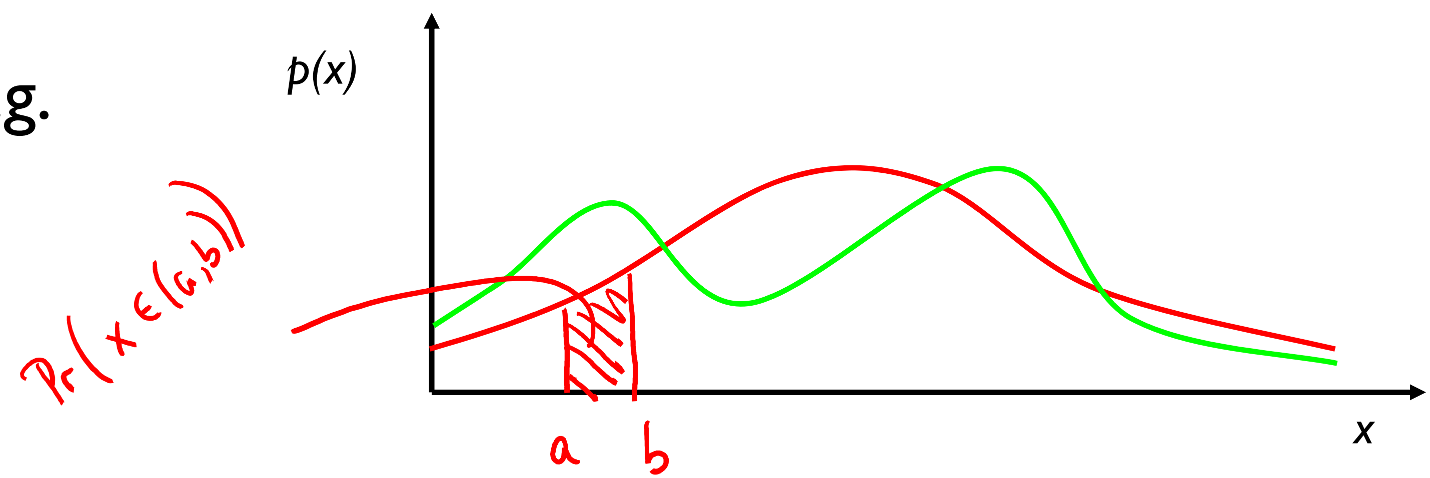
Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a probability density function.

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$

$\Pr(x=a)=0$

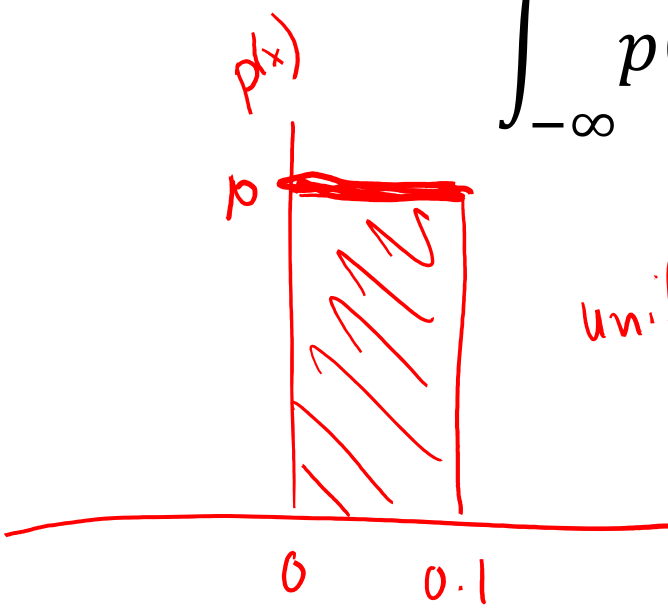
- E.g.



Continuous Random Variables

- unlike probabilities and probability mass functions, a probability density function can take on values greater than 1
  - e.g., uniform distribution over the range [0, 0.1]
- however, it is the case that

$$\int_{-\infty}^{\infty} p(x)dx = 1$$



uniform dist. over [0, 0.1]

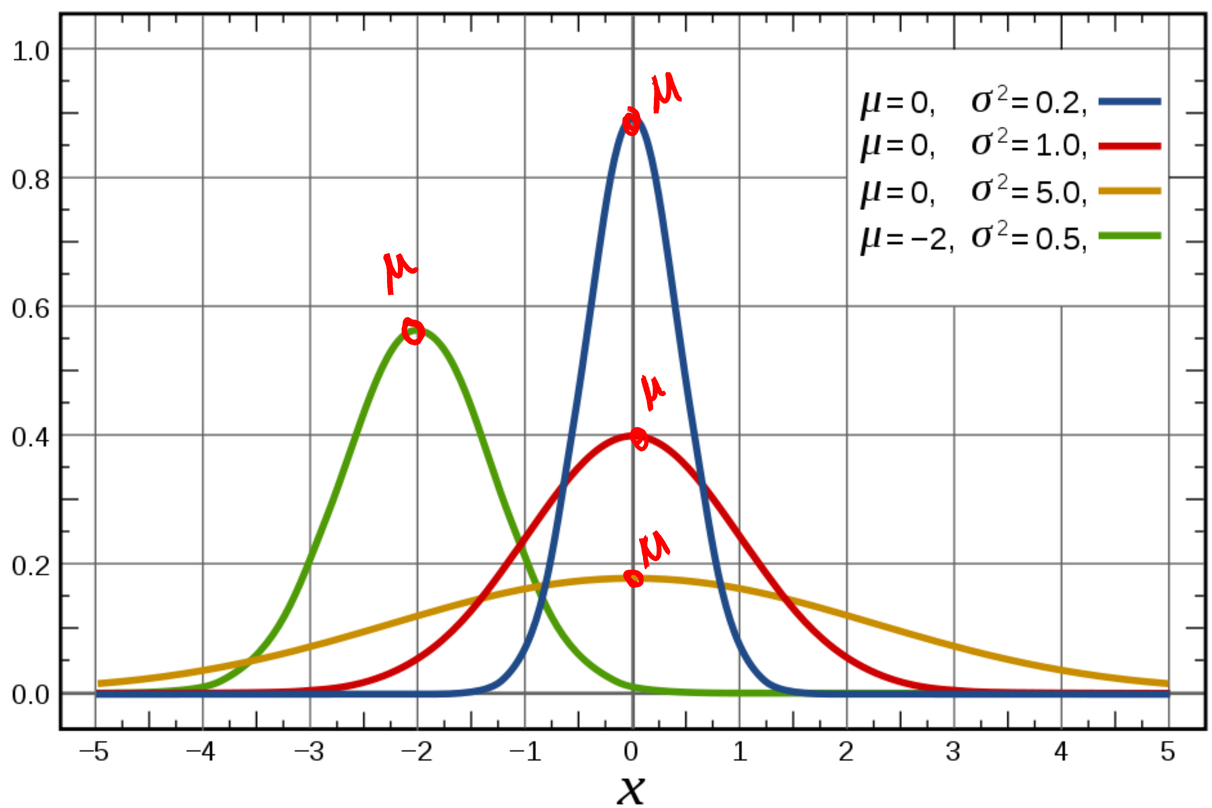
Continuous Random Variables

- normal or Gaussian distribution in 1D

mean

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

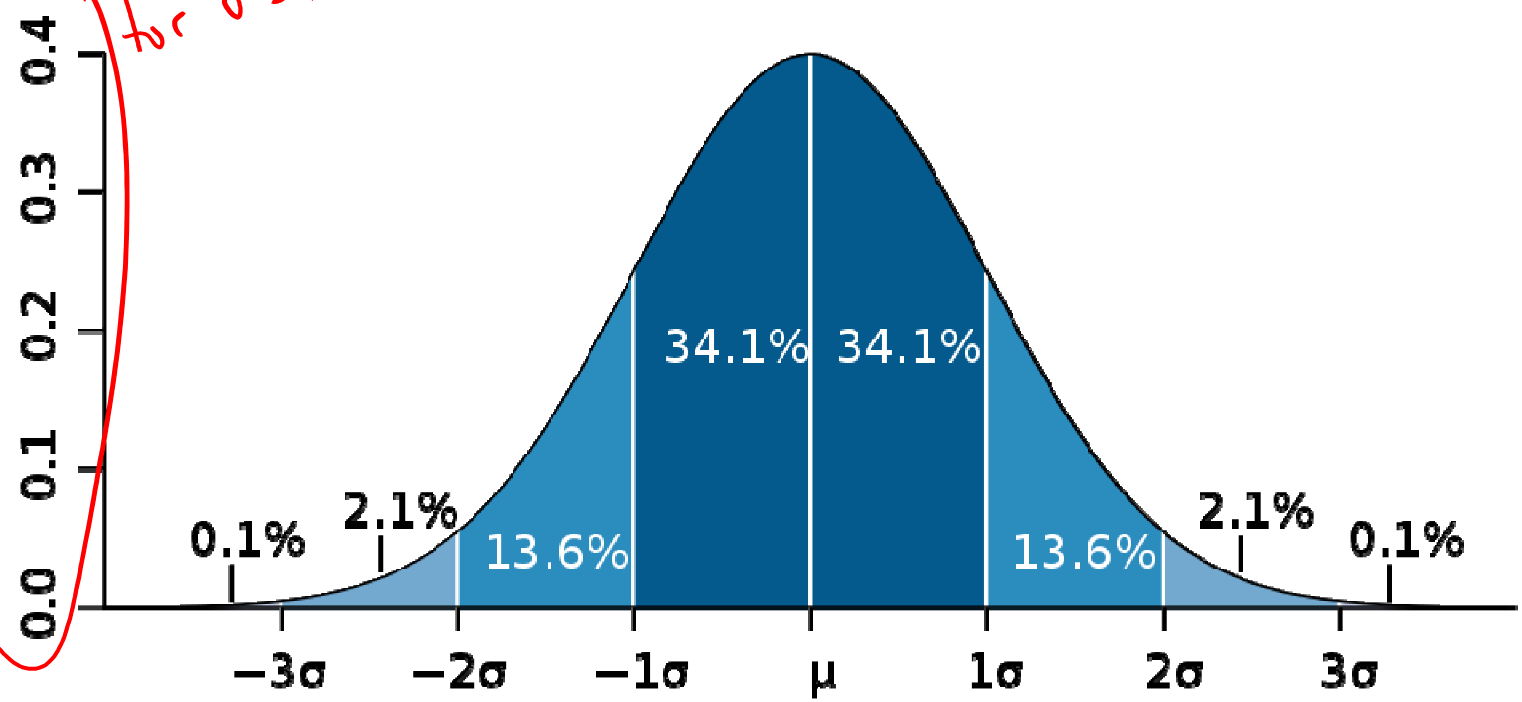
variance, σ std. deviation



Continuous Random Variables

- $1D$ normal, or Gaussian, distribution
  - μ mean
  - σ standard deviation
  - $\Sigma = \sigma^2$ variance

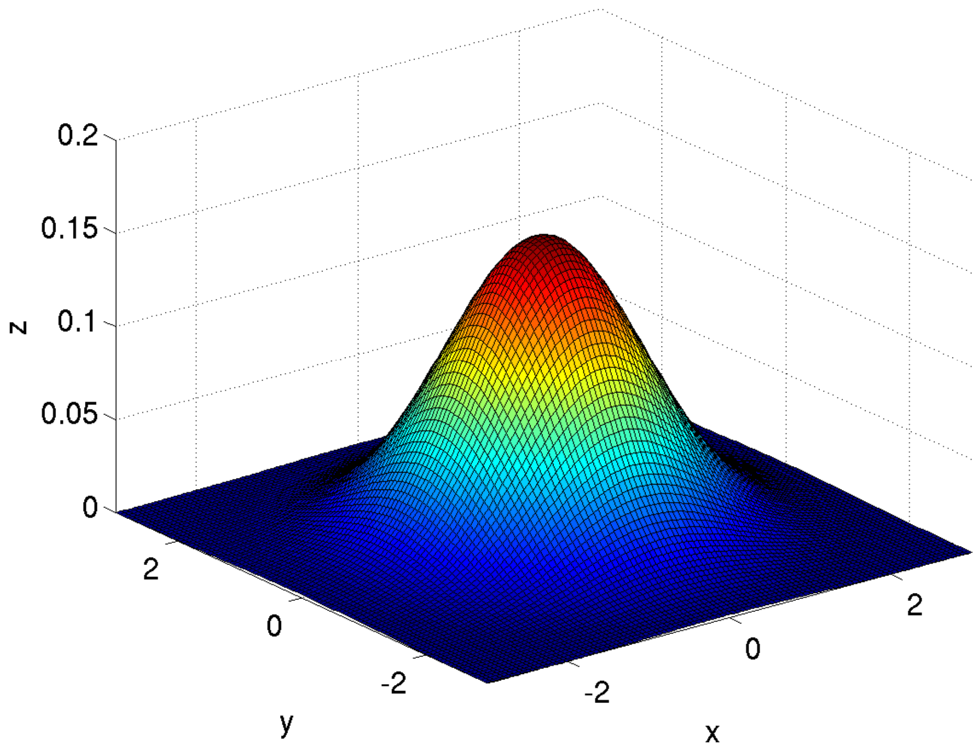
for $\sigma = 1$



Continuous Random Variables

- $2D$ normal, or Gaussian, distribution
  - μ mean
  - Σ covariance matrix

$$p(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$



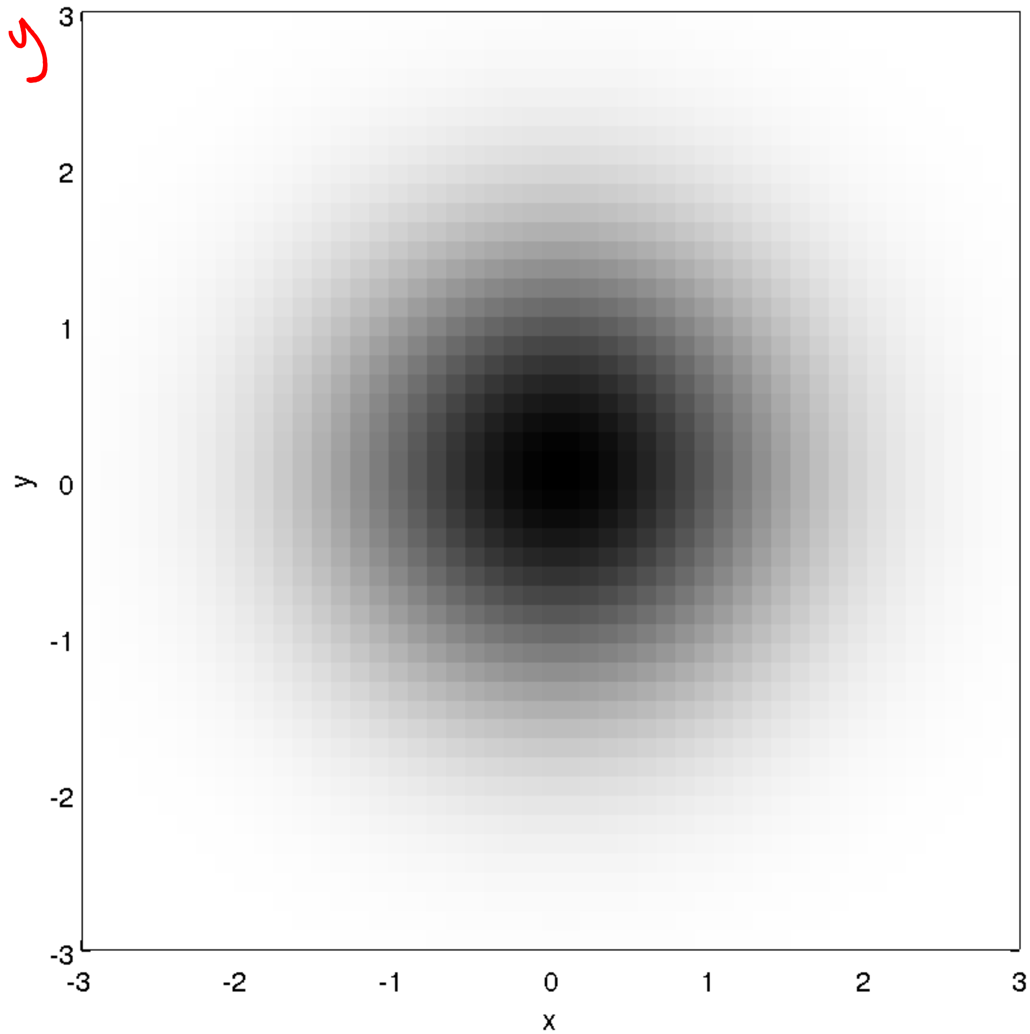
Continuous Random Variables

looking down on density function

- in $2D$
  - isotropic ✓ same in both x and y

$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



- darker values indicate larger values in density function

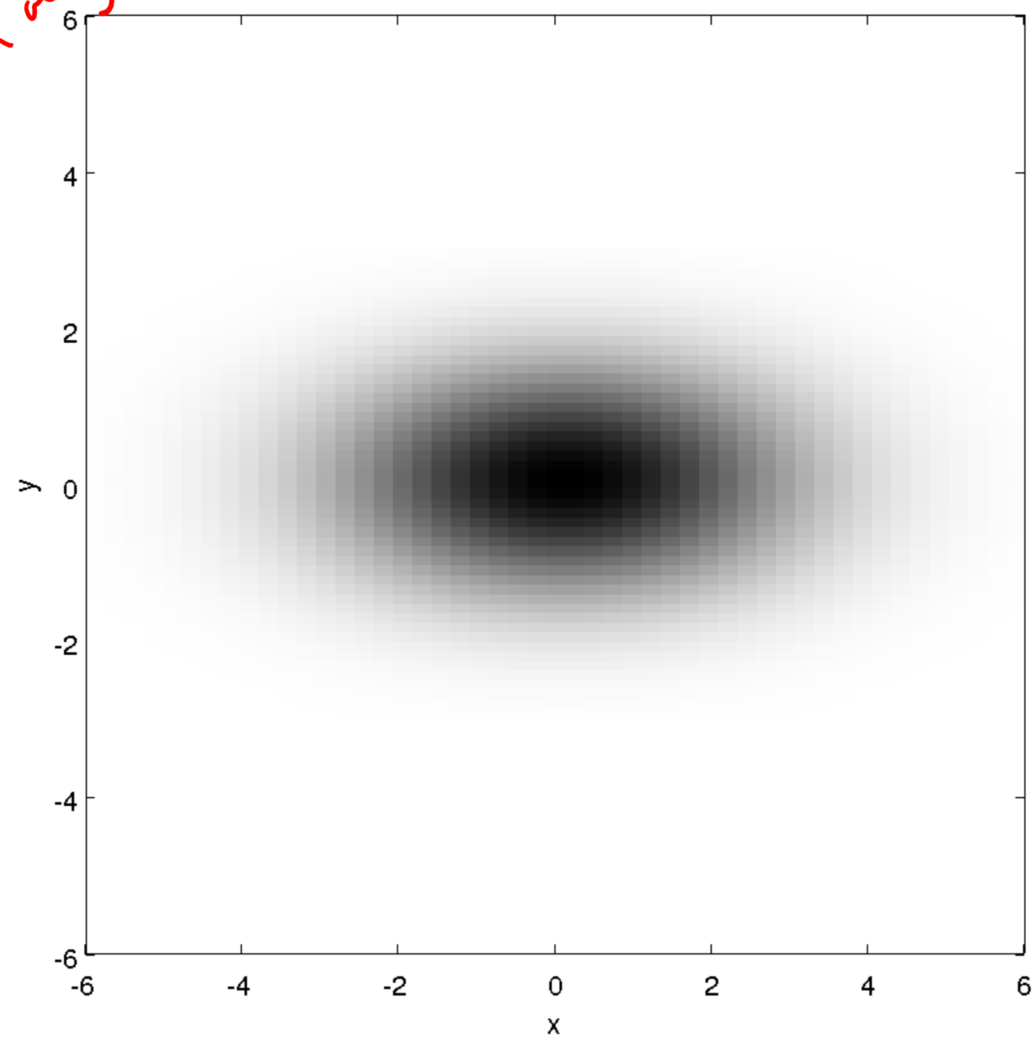
Continuous Random Variables

- in $2D$
  - anisotropic ✓

different in x and y

$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

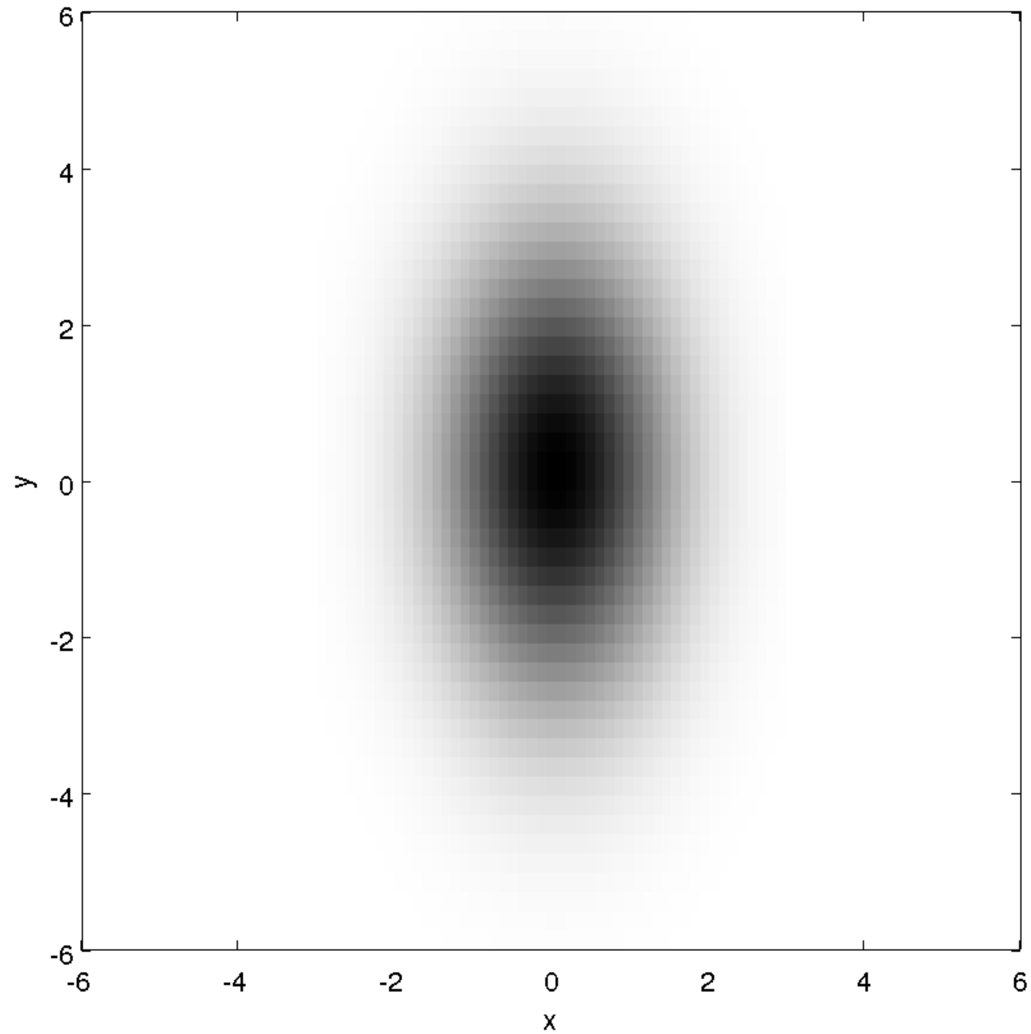


Continuous Random Variables

- in $2D$
  - anisotropic

$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

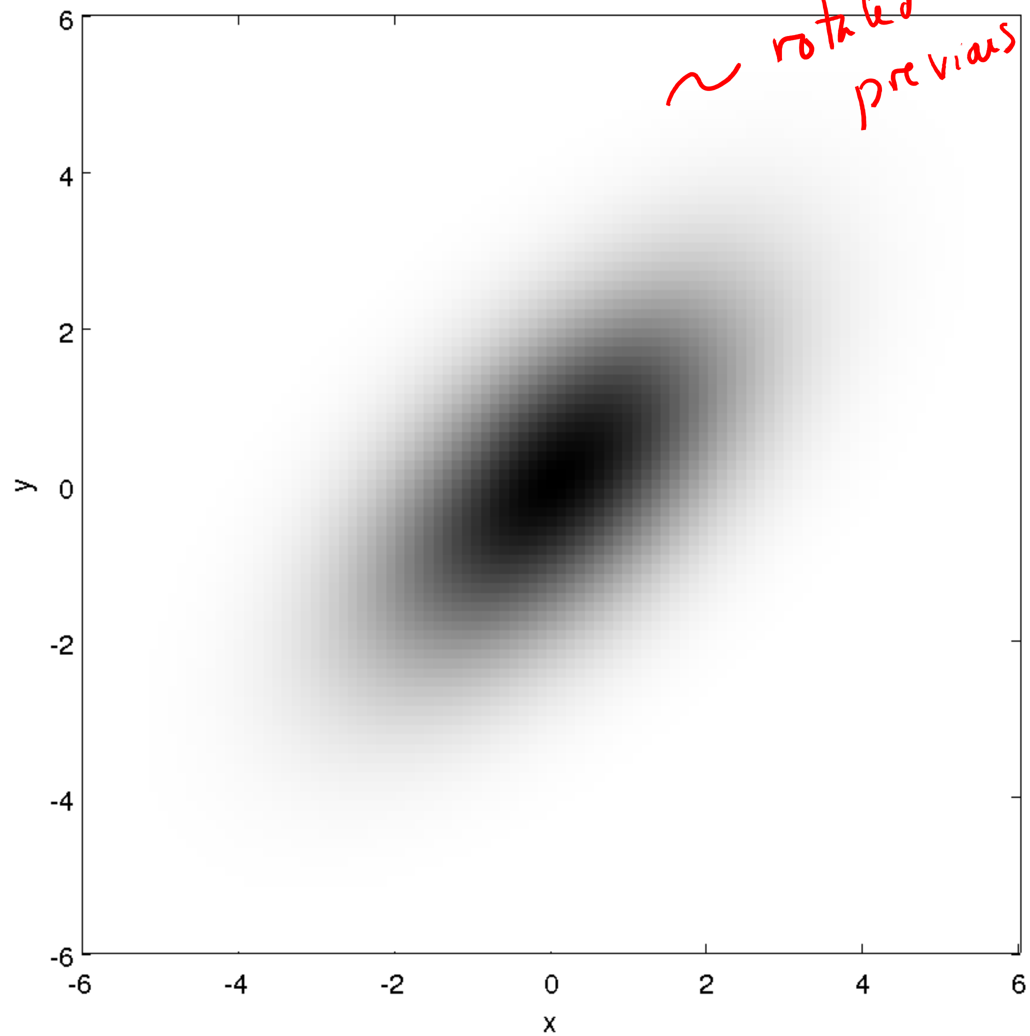


Continuous Random Variables

- in $2D$
  - anisotropic

$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Sigma = \underbrace{\begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}}_{\text{symmetric}}$$



~ rotated version of previous two slides

Covariance matrices

- the covariance matrix is always symmetric and positive semi-definite
- positive semi-definite:

$$x^T \Sigma x \geq 0 \text{ for all } x$$

- positive semi-definiteness guarantees that the eigenvalues of Σ are all greater than or equal to 0

sigma $= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

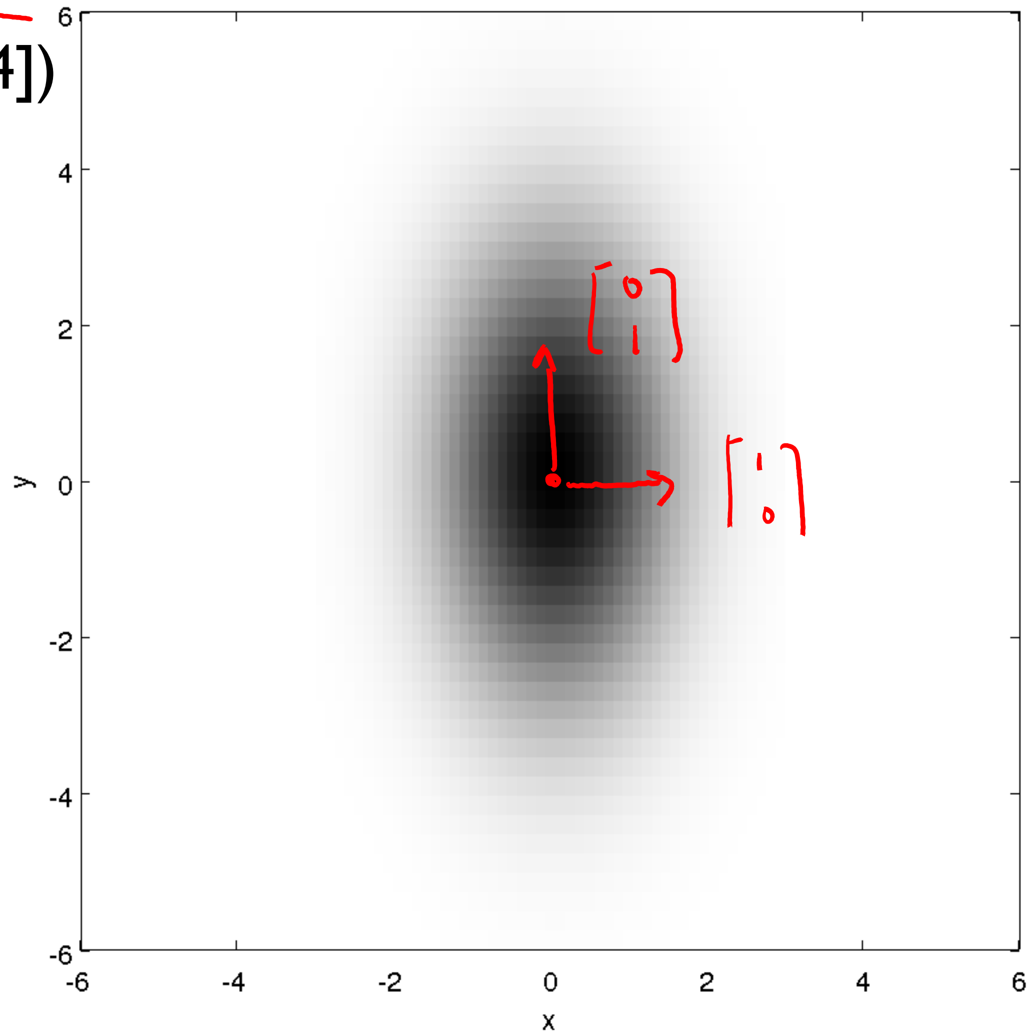
>> [v, d] = eig($\overbrace{\text{([1 0; 0 4])}}^{\text{sigma}}$)

v = eigenvectors

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

d = eigenvalues

$$\begin{matrix} 1 & 0 \\ 0 & 4 \end{matrix}$$



$$\begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$$

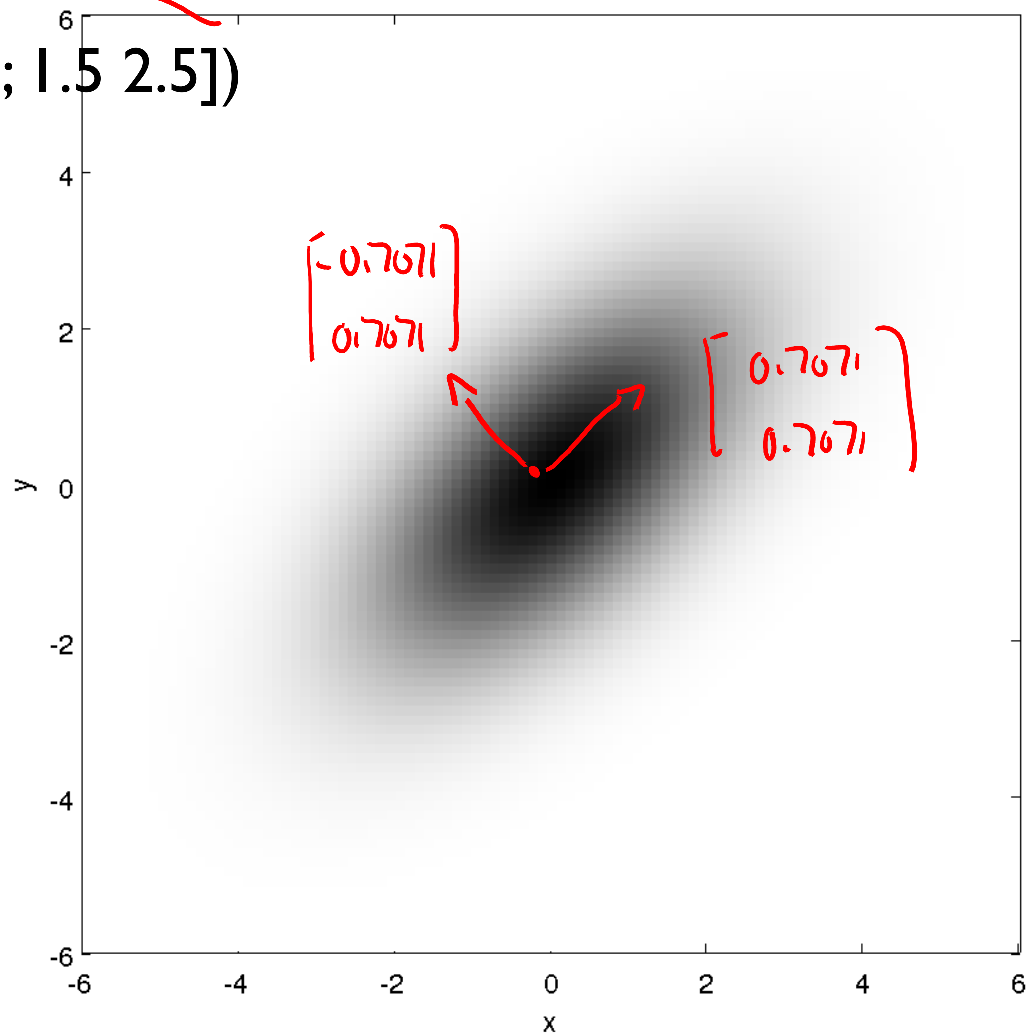
>> [v, d] = eig([2.5 1.5; 1.5 2.5])

v = eigenvectors

$$\begin{matrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{matrix}$$

d = eigenvalues

$$\begin{matrix} 1 & 0 \\ 0 & 4 \end{matrix}$$



Joint Probability

- the joint probability distribution of two random variables

$$P(X=x \text{ and } Y=y) = P(x,y)$$

describes the probability of the event that X has the value x and Y has the value y

- If X and Y are independent then

$$P(x,y) = P(x)\,P(y)$$

Joint Probability

- the joint probability distribution of two random variables

$$P(X = x \text{ and } Y = y) = P(x, y)$$

describes the probability of the event that X has the value x and Y has the value y

- example: two fair dice

$P(X = \text{even}) = \frac{3}{6}$ $P(Y = \text{even}) = \frac{3}{6}$

$$P(\overbrace{X = \text{even}} \text{ and } \overbrace{Y = \text{even}}) = 9/36$$

$$P(\underbrace{X = 1}_{\frac{1}{6}} \text{ and } \underbrace{Y = \text{not } 1}_{\frac{5}{6}}) = 5/36$$

Joint Probability

- example: insurance policy deductibles

$P(X=100)\,P(Y=0)$
$= (0.5)(0.25)$
$= 0.125$

		y		
		\$0	**\$100**	**\$200**
x	**\$100**	0.20	0.10	0.20
	\$250	0.05	0.15	0.30

← home (y)

↑ automobile (x)

$P(X=100) = 0.2 + 0.1 + 0.2$
$= 0.5$

$P(X=100 \text{ and } Y=0)$
$= 0.2$

$P(Y=0) = 0.2 + 0.05$
$= 0.25$

Joint Probability and Independence

- X and Y are said to be independent if

$$P(x,y) = P(x)\,P(y)$$

  for all possible values of x and y

- example: two fair dice

$$P(X\text{=even and }Y\text{=even}) = (1/2)\,(1/2)$$
$$P(X\text{=1 and }Y\text{=not 1}) = (1/6)\,(5/6)$$

- are X and Y independent in the insurance deductible example?

  - no, see previous slide

Marginal Probabilities

- the marginal probability distribution of X

$$P_X(x) = \sum_y P(x, y)$$

describes the probability of the event that X has the value x

- similarly, the marginal probability distribution of Y

$$P_Y(y) = \sum_x P(x, y)$$

describes the probability of the event that Y has the value y

Joint Probability

- example: insurance policy deductibles

y

x		$0	$100	$200
		$0	**$100**	**$200**
	$100	0.20	0.10	0.20
	$250	0.05	0.15	0.30

← home

↑ automobile

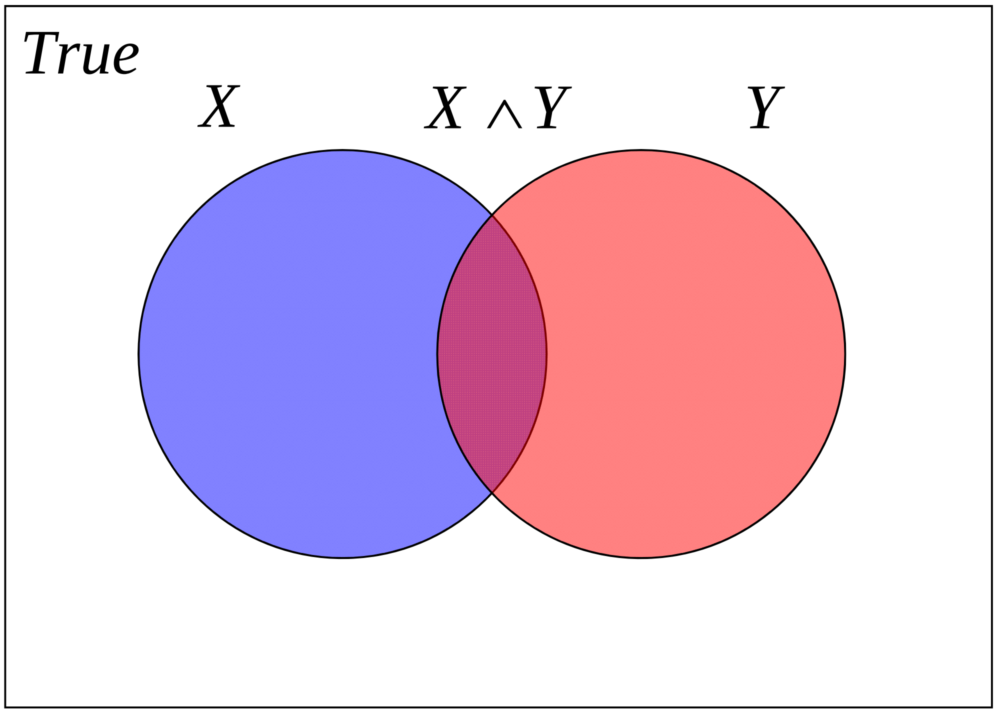
$$P(y=0) = \sum_x P(x,y)$$

$$= 0.2 + 0.05$$

$$= 0.25$$

Conditional Probability

- the conditional probability $P(x \mid y) = P(X{=}x \mid Y{=}y)$ is the probability of $P(X{=}x)$ if $Y{=}y$ is known to be true
  - "conditional probability of x given y"



$$p(x|y) = \frac{p(x \wedge y)}{p(y)}$$

Conditional Probability

$P(S) = 1$

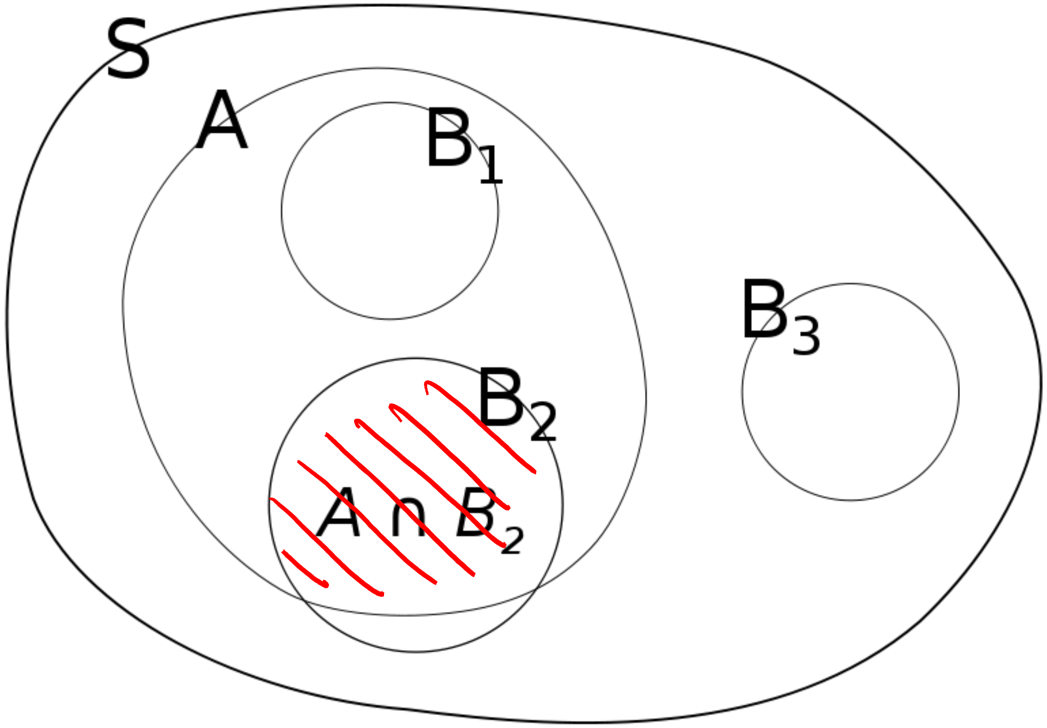
$P(A) \approx 0.3$

$P(A \mid B_3) = ?$ 0

$P(A \mid B_1) = ?$ 1

$P(A \mid B_2) = ?$

$$\frac{P(A \cap B_2)}{P(B_2)}$$



Conditional Probability

- "information changes probabilities"
- example:
  - roll a fair die; what is the probability that the number is a 3?

    $1/6$

  - what is the probability that the number is a 3 if someone tells you that the number is odd? is even?

    $\underbrace{\text{is odd?}}_{1/3}$ $\underbrace{\text{is even?}}_{0}$

Conditional Probability

- "information changes probabilities"
- example:
  - pick a playing card from a standard deck; what is the probability that it is the ace of hearts?

    $1/52$

  - what is the probability that it is the ace of hearts if someone tells you that it is an ace? that is a heart? that it is a king?

    $1/4$ $1/13$ 0

Conditional Probability

$$P(x \mid y) = \frac{P(x, y)}{P(y)}$$

- if X and Y are independent then

$$P(x, y) = P(x)P(y)$$

$$\therefore P(x \mid y) = \frac{P(x)P(y)}{P(y)} = P(x)$$

i.e. information about y does not affect $p(x/y)$

Bayes Formula

$p(x,y) = p(y,x)$

$$P(x,y) = \underbrace{P(x \mid y)P(y)}_{\text{from previous slide}} = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{\underbrace{P(x \mid y)}_{\text{posterior}} = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Back to Kinematics

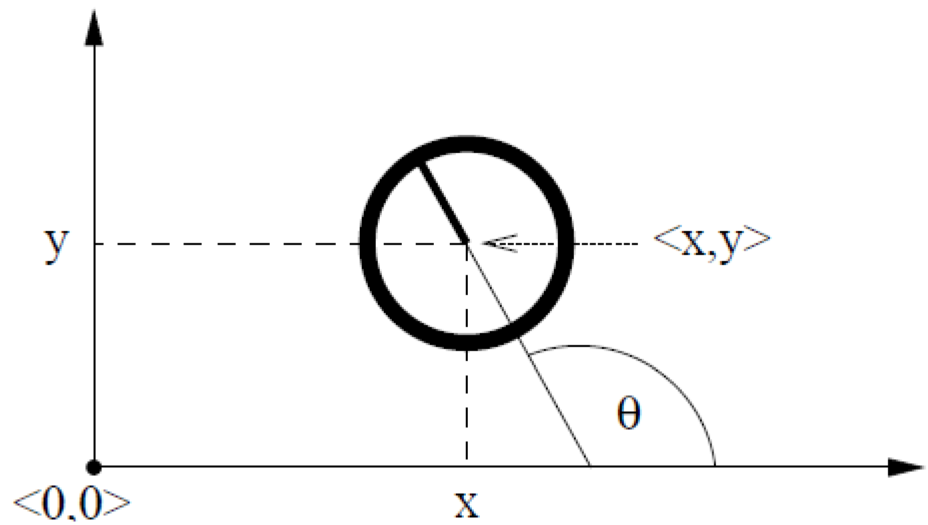


Figure 5.1 Robot pose, shown in a global coordinate system.

$$\text{pose vector or state} \quad x_t = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{location (in world frame)} \\ \left.\begin{matrix} \\ \end{matrix}\right\} \text{bearing or heading} \end{matrix}$$

Probabilistic Robotics

- we seek the conditional density

$$p(x_t \mid u_t, x_{t-1})$$

- what is the density of the state

$$x_t$$

given the motion command

$$u_t \quad = \begin{bmatrix} v_L \\ v_R \end{bmatrix} ? \quad \text{for differential drive}$$

performed at

$$x_{t-1}$$

Probabilistic Robotics

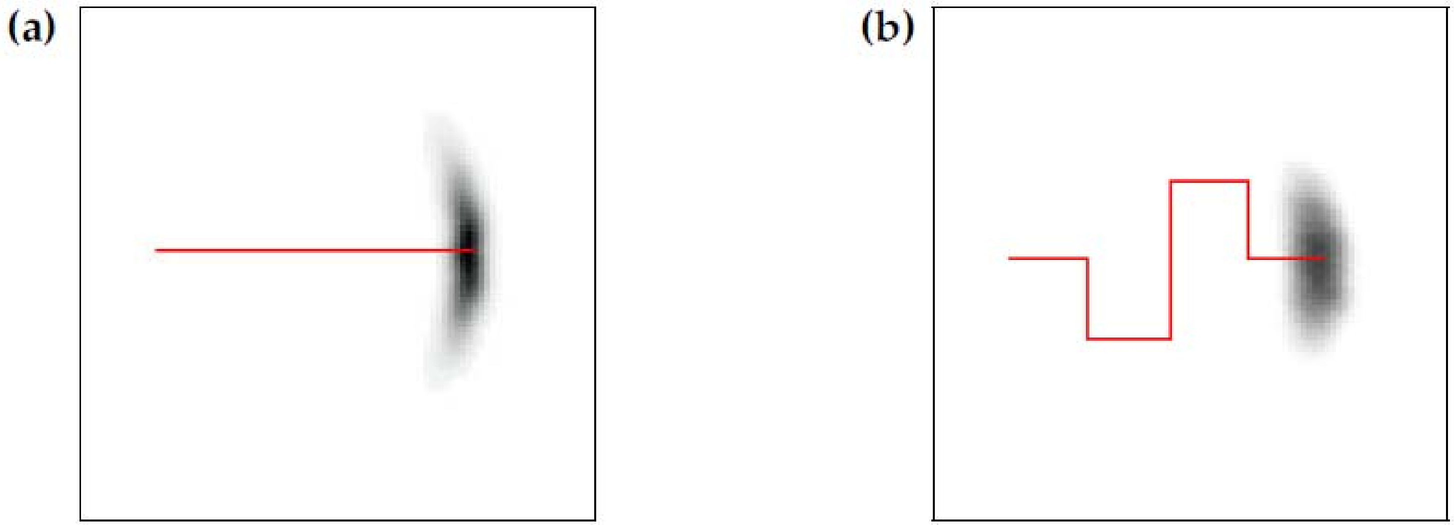


Figure 5.2 The motion model: Posterior distributions of the robot's pose upon executing the motion command illustrated by the solid line. The darker a location, the more likely it is. This plot has been projected into 2-D. The original density is three-dimensional, taking the robot's heading direction θ into account.

Velocity Motion Model

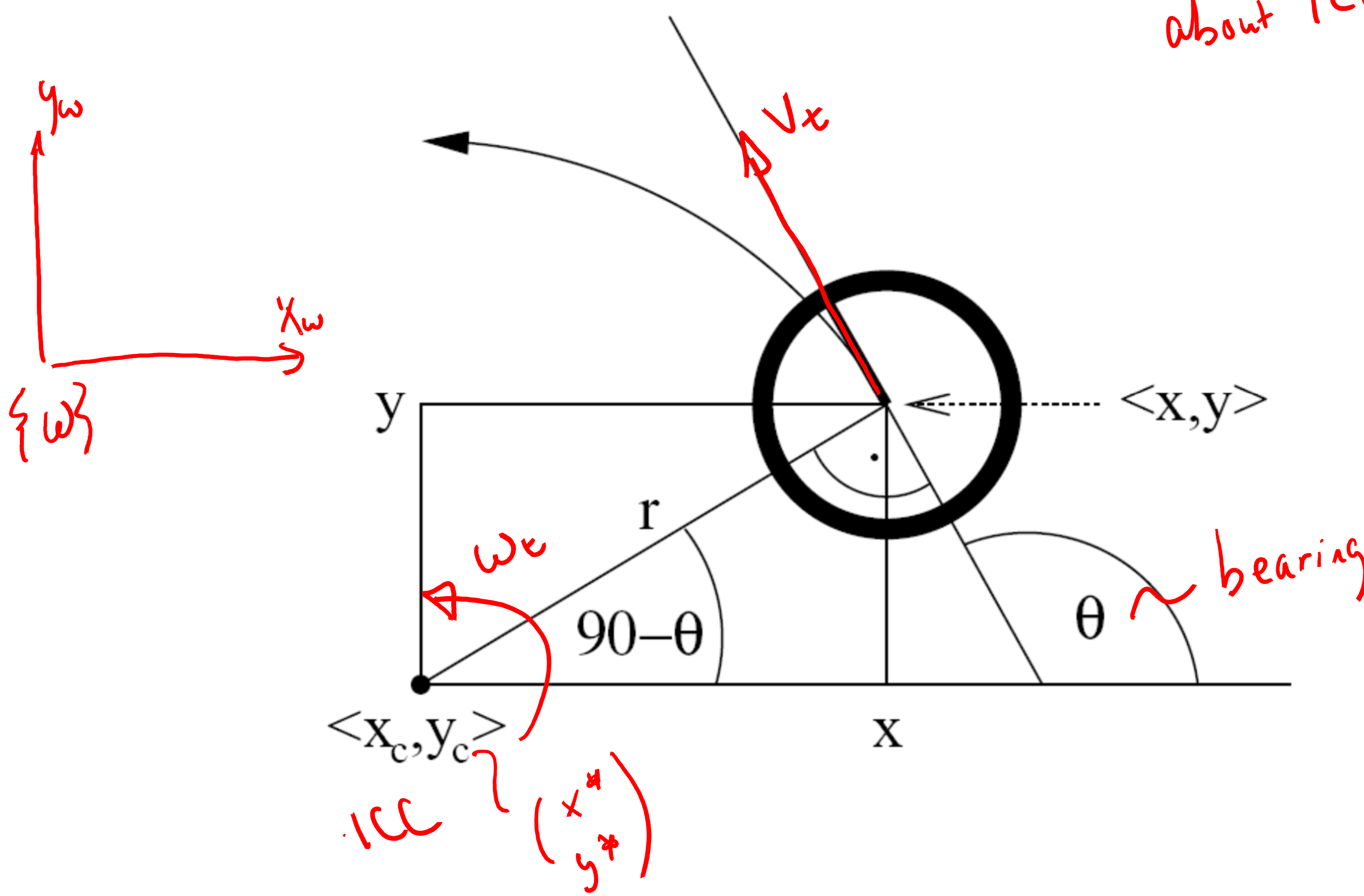
- assumes the robot can be controlled through two velocities
  - translational velocity v
  - rotational velocity ω (about an ICC)
- our motion command, or control vector, is

$$u_t = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix}$$

- positive values correspond to forward translation and counterclockwise rotation

Velocity Motion Model

circular motion about ICC



Velocity Motion Model

- center of circle

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin\theta \\ \lambda \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y-y') \\ \frac{y+y'}{2} + \mu(x'-x) \end{pmatrix}$$

where

$$\mu = \frac{1}{2} \frac{(x-x')\cos\theta + (y-y')\sin\theta}{(y-y')\cos\theta - (x-x')\sin\theta}$$

Velocity Motion Model

computes $p(x_t \mid u_t, x_{t-1})$

Algorithm motion_model_velocity(x_t, u_t, x_{t-1}):

new state

control vector $\begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$

old state

$$\mu = \frac{1}{2} \frac{(x - x')\cos\theta + (y - y')\sin\theta}{(y - y')\cos\theta - (x - x')\sin\theta}$$

$$x^* = \frac{x + x'}{2} + \mu(y - y')$$

$$y^* = \frac{y + y'}{2} + \mu(x' - x)$$

$$r^* = \sqrt{(x - x^*)^2 + (y - y^*)^2}$$

$$\Delta\theta = \mathrm{atan2}(y' - y^*, x' - x^*) - \mathrm{atan2}(y - y^*, x - x^*)$$

$$\hat{v} = \frac{\Delta\theta}{\Delta t}\, r^*$$

$$\hat{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\hat{\gamma} = \frac{\theta' - \theta}{\Delta t} - \hat{\omega}$$

return $\mathbf{prob}(v - \hat{v}, \alpha_1|v| + \alpha_2|\omega|) \cdot \mathbf{prob}(\omega - \hat{\omega}, \alpha_3|v| + \alpha_4|\omega|) \cdot \mathbf{prob}(\hat{\gamma}, \alpha_5|v| + \alpha_6|\omega|)$